CLASSIFICATION OF NEGATIVELY PINCHED MANIFOLDS WITH AMENABLE FUNDAMENTAL GROUPS

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Abstract. We give a diffeomorphism classification of pinched negatively curved manifolds with amenable fundamental groups, namely, they are precisely the Möbius band, and the products of $\mathbb{R}$ with the total spaces of flat vector bundles over closed infranilmanifolds.

1. Introduction

In this paper we study manifolds of the form X/Γ, where X is a simply-connected complete Riemannian manifold with sectional curvatures pinched (i.e. bounded) between two negative constants, and Γ is a discrete torsion free subgroup of the isometry group of X. According to [BS87], if Γ is amenable, then either Γ stabilizes a biinfinite geodesic, or else Γ fixes a unique point z at infinity. The case when Γ stabilizes a biinfinite geodesic is completely understood, namely the normal exponential map to the geodesic is a Γ-equivariant diffeomorphism, hence X/Γ is a vector bundle over S^1; there are only two such bundles each admitting a complete hyperbolic metric.

If Γ fixes a unique point z at infinity (such groups are called *parabolic*), then Γ stabilizes horospheres centered at z and permutes geodesics asymptotic to z, so that given a horosphere H, the manifold X/Γ is diffeomorphic to the product of H/Γ with $\mathbb{R}$. We refer to H/Γ as a *horosphere quotient*. In this case a delicate result of B. Bowditch [Bow93] shows that Γ must be finitely generated, which by Margulis lemma [BGS85] implies that Γ is virtually nilpotent.

The main result of this paper is a diffeomorphism classification of horosphere quotients, namely we show that, up to a diffeomorphism, the classes of horosphere quotients and (possibly noncompact) infranilmanifolds coincide.

By an *infranilmanifold* we mean the quotient of a simply-connected nilpotent Lie group G by the action of a torsion free discrete subgroup Γ of the semidirect product of G with a compact subgroup of $\mathrm{Aut}(G)$.

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Theorem 1.1. *For a smooth manifold N the following are equivalent:*

(1) *N is a horosphere quotient;*
(2) *N is diffeomorphic to an infranilmanifold;*
(3) *N is the total space of a flat Euclidean vector bundle over a compact infranilmanifold.*

The implication $(3) \Rightarrow (2)$ is straightforward, $(2) \Rightarrow (1)$ is proved by constructing an explicit warped product metric of pinched negative curvature. The proof of $(1) \Rightarrow (3)$ occupies most of the paper, and depends on the collapsing theory of J. Cheeger, K. Fukaya, and M. Gromov [CFG92].

If N is compact (in which case the conditions (2), (3) are identical), the implication $(1) \Rightarrow (2)$ follows from Gromov's classification of almost flat manifolds, as improved by E. Ruh, while the implication $(2) \Rightarrow (1)$ is new. If N is noncompact, Theorem 1.1 is nontrivial even when $\pi_1(N) \cong \mathbb{Z}$, although the proof does simplify in this case. A direct algebraic proof of $(2) \Rightarrow (3)$ was given in [Wil00b, Theorem 6], but the case when N is a nilmanifold was already treated in [Mal49], where it is shown that any nilmanifold is diffeomorphic to the product of a compact nilmanifold and a Euclidean space.

We postpone the discussion of the proof till Section 2, and just mention that the proof also gives geometric information about horosphere quotients, e.g. we show that H/Γ is diffeomorphic to a tubular neighborhood of some orbit of an N-structure on H/Γ.

By Chern-Weil theory any flat Euclidean vector bundle has zero rational Euler and Pontrjagin classes. Moreover, by [Wil00a] any flat Euclidean bundle with virtually abelian holonomy is isomorphic to a bundle with finite structure group. Thus the vector bundle in (3) becomes trivial in a finite cover, and has zero rational Euler and Pontrjagin classes, and in particular, any horosphere quotient is finitely covered by the product of a compact nilmanifold and a Euclidean space.

Corollary 1.2. *A smooth manifold M with amenable fundamental group admits a complete metric of pinched negative curvature if and only if it is diffeomorphic to the Möbius band, or to the product of a line and the total space a flat Euclidean vector bundle over a compact infranilmanifold.*

The pinched negative curvature assumption in Corollary 1.2 cannot be relaxed to $-1 \leq \sec \leq 0$ or $\sec \leq -1$, e.g. because these assumptions do not force the fundamental group to be virtually nilpotent [Bow93, Section 6]. More delicate examples come from the work of M. Anderson [And87] who proved that each vector bundle over a closed nonpositively curved manifold (e.g. a torus) carries a complete Riemannian metric with $-1 \leq \sec \leq 0$. Since in each dimension there are only finitely many isomorphism classes of flat Euclidean bundles over

a given compact manifold, all but finitely many vector bundles over tori admit no metrics of pinched negative curvature. Also $-1 \leq \sec(M) \leq 0$ can be turned into $\sec(M \times \mathbb{R}) \leq -1$ for the warped product metric on $M \times \mathbb{R}$ with warping function e^t [BO69], hence Anderson's examples carry metrics with $\sec \leq -1$ after taking product with $\mathbb{R}$. Specifically, if E is the total space of a vector bundle over a torus with nontrivial rational Pontrjagin class, then $M = E \times \mathbb{R}$ carries a complete metric of $\sec \leq -1$ but not of pinched negative curvature. Finally, Anderson also showed that every vector bundle over a closed negatively curved manifold admits a complete Riemannian metric of pinched negative curvature, hence amenability of the fundamental group is indispensable.

Because an infranilmanifold with virtually abelian fundamental group is flat, Theorem 1.1 immediately implies the following.

Corollary 1.3. *Let M be a smooth manifold with virtually abelian fundamental group. Then the following are equivalent*
(i) *M admits a complete metric of* $\sec \equiv -1$*;*
(ii) *M admits a complete metric of pinched negative curvature.*

In [Bow95] Bowditch developed several equivalent definitions of geometrical finiteness for pinched negatively curved manifolds, and conjectured the following result.

Corollary 1.4. *Any geometrically finite pinched negatively curved manifold X/Γ is diffeomorphic to the interior of a compact manifold with boundary.*

We believe that the main results of this paper, including Corollary 1.4, should extend to the orbifold case, i.e. when Γ is not assumed to be torsion free. However, working in the orbifold category creates various technical difficulties, both mathematical and expository, and we do not attempt to treat the orbifold case in this paper.

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Contents

2. Sketch of the proof of (1) $\Rightarrow$ (3)

The Busemann function corresponding to z gives rise to a C^2-Riemannian submersion $X/\Gamma \to \mathbb{R}$ whose fibers are horosphere quotients each equipped with the induced C^1-Riemannian metric g_t. By the Rauch comparison theorem the second fundamental form of a horosphere is bounded in terms of curvature bounds of X (cf. [BK84]). In particular, each fiber has curvature uniformly bounded above and below in comparison sense. Let $\sigma(t)$ be a horizontal geodesic in X/Γ, i.e. a geodesic that projects isometrically to $\mathbb{R}$. Because of the exponential convergence of geodesics in X, the manifold X/Γ is "collapsing" in the sense that the unit balls around $\sigma(t) \in X/\Gamma$ form an exhaustion of X/Γ and have small injectivity radius for large t. Similarly, each fiber of $X/\Gamma \to \mathbb{R}$ also collapses, and in fact X/Γ is noncollapsed in the direction transverse to the fibers.

There are essential difficulties in applying the collapsing theory of [CFG92] to X/Γ. First, we do not know whether $(X/\Gamma, \sigma(t))$ converges in pointed Gromov-Hausdorff topology to a single limit space. By general theory, the family $(X/\Gamma, \sigma(t))$ is precompact and thus has many converging subsequences.

While different limits might be non-isometric, one of the main steps of the proof is obtaining a uniform (i.e. independent of the subsequence) lower bound on the "injectivity radius" of the limit spaces at the base point. This is done by a comparison argument involving taking "almost square roots" of elements of Γ, and using the flat connection of [Bow93] discussed below. Another complication is that the N-structure on $(X/\Gamma, \sigma(t))$ provided by [CFG92] may well have zero-dimensional orbits outside the unit ball around $\sigma(t)$, in other words a large noncompact region of $(X/\Gamma, \sigma(t))$ may be non-collapsed, which makes it hard to control topology of the region.

However, once the "injectivity radius" bound is established, critical points of distance functions considerations yield the "product structure at infinity" for X/Γ, and also for $(H/\Gamma, g_t)$ if t is large enough.

Furthermore, one can show that H/Γ is diffeomorphic to the normal bundle of an orbit O_t of the N-structure. The orbit corresponds to the point in a limit space given to us by the convergence, and at which we get an "injectivity radius" bound. This depends on a few results on Alexandrov spaces with curvature bounded below, with key ingredients provided by [Kap02] and [PP].

By the collapsing theory, the structure group of the normal bundle to the orbit of an N-structure is a finite extension of a torus group [CFG92]. Of course, not every such a bundle has a flat Euclidean structure.

The flatness of the normal bundle to the orbit is proved using a remarkable flat connection discovered by B. Bowditch [Bow93], and later in a different disguise by W. Ballmann and J. Brüning [BB01], who were apparently unaware of [Bow93]. It follows from [Bow93] or [BB01] that each pinched negatively curved manifold X/Γ, where Γ fixes a unique point z at infinity, admits a natural flat C^0-connection that is compatible with the metric and has nonzero torsion, and such that on short loops it is close to the Levi-Civita connection. Furthermore, the parallel transport of the connection preserves the fibration of X/Γ by horosphere quotients. Hence each horosphere quotient has flat tangent bundle.

In fact, we prove a finer result that the normal bundle to O_t is also flat, for suitable large t. (Purely topological considerations are useless here since there exist vector bundles without flat Euclidean structure whose total spaces have flat Euclidean tangent bundles, for example this happens for any nontrivial orientable $\mathbb{R}^2$-bundle over the 2-torus that has even Euler number). It turns out that O_t sits with flat normal bundle in a totally geodesic stratum of the N-structure, so it suffices to show that the normal bundle to the stratum is flat, when restricted to O_t. Now since the above flat connection is close to Levi-Civita connection the normal bundle is "almost flat", and it can be made flat by averaging via center of mass. This completes the proof.

Throughout the proof we use the collapsing theory developed in [CFG92]. This paper is based on the earlier extensive work of Fukaya, and Cheeger-Gromov, and many arguments in [CFG92] are merely sketched. We suggest reading [Fuk90] for a snapshot of the state of affairs before [CFG92], and [PT99, PRT99, FR02] for a current point of view.

3. Topological digression

The result of Bowditch [Bow93] that horosphere quotients have finitely generated fundamental groups actually implies that any horosphere quotient is homotopy equivalent to a compact infranilmanifold (because any torsion free finitely generated virtually nilpotent group is the fundamental group of a compact infranilmanifold [Dek96], and because for aspherical manifolds any π_1-isomorphism is induced by a homotopy equivalence).

To help appreciate the difference between this statement and Theorem 1.1, we discuss several types of examples that are allowed by Bowditch's result, and are ruled out by Theorem 1.1. The simplest example is a vector bundle over an infranilmanifold with nonzero rational Euler or Pontrjagin class: such a manifold cannot be the total space of a flat Euclidean bundle, as is easy to see using the fact that the tangent bundle to any infranilmanifold is flat.

Another example is the product of a closed infranilmanifold and a contractible manifold of dimension > 2 that is not simply-connected at infinity. Finally, even more sophisticated examples come from the fact that below metastable range (starting at which any homotopy equivalence is homotopic to a smooth embedding, by Haefliger's embedding theorem) there are many smooth manifolds that are thickenings of say a torus, yet are not vector bundles over the torus. It would be interesting to see whether the "weird" topological constructions of this paragraph can be realized geometrically, even as nonpositively curved manifolds.

4. Parallel transport through infinity and rotation homomorphism

Let X be a simply-connected complete pinched negatively curved n-manifold normalized so that $-a^2 \leq sec(X) \leq -1$. One of the key properties of X used in this section is that any two geodesic rays in X that are asymptotic to the same point at infinity converge exponentially, i.e. for any asymptotic rays $\gamma_1(t), \gamma_2(t)$ with $\gamma_1(0), \gamma_2(0)$ lying on the same horosphere, the function $d(\gamma_1(t), \gamma_2(t))$ is monotonically decreasing as $t \to \infty$ and

$$e^{-at}c_1(a, d(\gamma_1(0), \gamma_2(0))) \leq d(\gamma_1(t), \gamma_2(t)) \leq e^{-t}c_2(a, d(\gamma_1(0), \gamma_2(0))).$$

where $c_i(a,d)$ is linear in d for small d. This is proved by triangle comparison with spaces of constant negative curvature.

Bowditch introduced a connection on X that we now describe (see [Bow93, Section 3] for details). Fix a point z at infinity of X. Let $w_i \to z$ as $i \to \infty$. For any $x, y \in X$, consider the parallel transport map from x to w_i followed by the parallel transport from w_i to y along the shortest geodesics. This defines an isometry between the tangent spaces at x and y. By [Bow93, Lemma 3.1], this map converges to a well-defined limit isometry $P^\infty_{xy}\colon T_xM \to T_yM$ as $i \to \infty$. We refer to P^∞_{xy} as the *parallel transport through infinity from x to y*.

We denote the Levi-Civita parallel transport from x to y along the shortest geodesic by P_{xy}; clearly, if x, y lie on a geodesic ray that ends at z, then $P^\infty_{xy} = P_{xy}$. A key feature of P^∞ is that it approximates the Levi-Civita parallel transport on short geodesic segments (see [Bow93, Lemma 3.2]; more details can be found in [BK81, Section 6]). This is because any geodesic triangle in X spans a "ruled" surface of area at most the area of the comparison triangle in the hyperbolic plane of $\sec = -1$. By exponential convergence of geodesics the area of the comparison triangle is bounded above by a constant times the shortest side of the triangle. As the holonomy around the circumference of the triangle is bounded by the integral of the curvature over its interior, we conclude that $|P_{xy} - P^\infty_{xy}| \le q(a)d(xy)$, where $q(a)$ is a constant depending only on a.

Given $x \in X$, fix an isometry $\mathbb{R}^n \to T_xX$ and translate it around X using P^∞. This defines a P^∞-invariant trivialization of the tangent bundle to X. Let $\mathrm{Iso}_z(X)$ be the group of isometries of X that fixes z. For any point $y \in X$ look at the map $\mathrm{Iso}_z(X) \to O(n)$ given by $\gamma \mapsto P^\infty_{\gamma(y)y} \circ d\gamma$. It turns out that this map is a homomorphism independent of y. We call it the *rotation homomorphism*. Starting with a different base point $x \in X$ or a different isometry $\mathbb{R}^n \to T_xX$ has the effect of replacing the rotation homomorphism by its conjugate.

Now if Γ is a discrete torsion free subgroup of $\mathrm{Iso}_z(X)$, then, since the rotation homomorphism is independent of y, P^∞ gives rise to a flat connection on X/Γ with holonomy given by the rotation homomorphism. By the above discussion of P^∞, this is a C^0 flat connection that is compatible with the metric and close to the Levi-Civita connection on short loops. Of course, this connection has torsion.

Remark 4.1. The above discussion is easily seen to be valid if X is a simply-connected complete C^1-Riemannian manifold of pinched negative curvature in the comparison sense. This is because any such C^1-metric can be approximated uniformly in C^1-topology by smooth Riemannian metrics of pinched negative curvature [Nik89], perhaps with slightly larger pinching. Then the distance

functions and Levi-Civita connections converge uniformly in C^0-topology, and we recover all the statements above.

Remark 4.2. The connection of Bowditch, that was described above, was reinvented later in a different disguise by Ballmann and Brüning [BB01, Section 3]. The connection in [BB01] is defined by an explicit local formula in terms of the curvature tensor and the Levi-Civita connection of X. Actually, [BB01] only discusses the case of compact horosphere quotients however all the arguments there are local, hence they apply to any horosphere quotient. The only feature which is special for compact horosphere quotients is that in that case the connection has finite holonomy group [BB01], as follows from estimates in [BK81]. For non-compact horosphere quotients, the holonomy need not be finite as seen by looking at a glide rotation with irrational angle in $\mathbb{R}^3$, thought of as a horosphere in the hyperbolic 4-space. We never have to use [BB01] in this paper, however, for completeness we discuss their construction in Appendix C, where we also show that the connections of [Bow93] and [BB01] coincide.

5. Passing to the limit

Let X be a simply-connected complete pinched negatively curved n-manifold normalized so that $-a^2 \leq sec(X) \leq -1$, let $c(t)$ be a biinfinite geodesic in X, and let Γ be a closed subgroup of $\mathrm{Iso}(X)$ that fixes the point $c(\infty)$ at infinity. We refer to the gradient flow b_t of a Busemann function for $c(t)$ as *Busemann flow*. Following Bowditch, we sometimes use the notation $x + t := b_t(x)$.

Since X has bounded curvature and infinite injectivity radius, the family $(X, c(t), \Gamma)$ has a subsequence $(X, c(t_i), \Gamma)$ that converges to (X_∞, p, G) in the equivariant pointed $C^{1,\alpha}$-topology [Pet98, Chapter 10]. Here X_∞ is a smooth manifold with $C^{1,\alpha}$-Riemannian metric that has infinite injectivity radius and the same curvature bounds as X in the comparison sense, and G is a closed subgroup of $\mathrm{Iso}(X_\infty)$. Note that $\mathrm{Iso}(X_\infty)$ is a Lie group that acts on X by C^3-diffeomorphisms (this last fact is probably known but for a lack of reference we give a simple proof in Appendix B).

Furthermore, geodesic rays in X that start at uniformly bounded distance from $c(t_i)$ converge to rays in X_∞. In particular, the rays $c(t + t_i)$ starting at $c(t_i)$ converge to a ray $c_\infty(t)$ in X_∞ that starts at p, and the corresponding Busemann functions also converge. Since the Busemann functions on X are C^2 [HIH77], they converge to a C^1 Busemann function on X_∞. Thus the horosphere passing through p is a C^1-submanifold of X_∞, and is the limit of horospheres passing through $c(t_i)$. The Busemann flow is C^1 on X, and C^0 on X_∞. Since the horospheres in X and X_∞ have the same dimension, the sequence of horospheres passing through $c(t_i)$ does not collapse, and more

generally, each horosphere centered at $c_\infty(\infty)$ is the limit of a noncollapsing sequence of horospheres in X.

It is easy to see that the group G fixes $c_\infty(\infty)$, i.e. any $\gamma \in G$ takes c_∞ to a ray asymptotic to c_∞. Furthermore, G leaves the horospheres corresponding to $c_\infty(\infty)$ invariant.

Thus, one can define the rotation homomorphism $\phi_\infty \colon G \to O(n)$ corresponding to the point $c_\infty(\infty)$. The point only determines ϕ_∞ up to conjugacy so we also need to fix an isometry $L \colon \mathbb{R}^n \to T_p X_\infty$. Similarly, a choice of an isometry $L_i \colon \mathbb{R}^n \to T_{c(t_i)} X$ specifies the rotation homomorphism $\phi_i \colon \Gamma \to O(n)$ corresponding to the point $c(\infty)$. We can assume that $\phi_i = \phi_0$ for each i, by choosing L_i equal to L_0 followed by the parallel transport $P^\infty_{c(t_0),c(t_i)} = P_{c(t_0),c(t_i)}$. Henceforth we denote ϕ_0 by ϕ. Also it is convenient to choose L as follows.

Lemma 5.1. *After passing to a subsequence of $(X, c(t_i), \Gamma)$, there exist L such that if $\gamma_i \to \gamma$, then $\phi(\gamma_i) \to \phi_\infty(\gamma)$.*

Proof. Since $(X, c(t_i), \Gamma) \to (X_\infty, p, G)$ in pointed equivariant $C^{1,\alpha}$ topology, we can find the corresponding $C^{1,\alpha}$ approximations $f_i \colon B(c(t_i), 1) \to B(p, 1)$. We may assume that $df_i \colon T_{c(t_i)} X \to T_p X_\infty$ is an isometry for all i. By compactness of $O(n)$, $df_i \circ L_i$ subconverge to an isometry $L \colon \mathbb{R}^n \to T_p X$, so by modifying f_i slightly we can assume that $df_i \circ L_i = L$. This L is then used to define ϕ_∞, and it remains to show that if $\gamma_i \to \gamma$, then $\phi(\gamma_i) \to \phi_\infty(\gamma)$. For the rest of the proof we suppress L_i, L.

Since $d\gamma_i \to d\gamma$ it is enough to show that parallel transports through infinity from $c(t_i)$ to $\gamma_i(c(t_i))$ converge to the parallel transport through infinity from p to $\gamma(p)$.

For $x \in X$ and $c(t)$ that lie on the same horosphere, and for any $s > t$, denote by $P_{x,s,c(t)} \colon T_{c(t)} M \to T_x M$ the parallel transport along the piecewise geodesic path $x, x+s, c(t)+s, c(t)$. Here $c(t)+s = c(t+s)$, $x+s$ also lie on the same horosphere. Now $|P^\infty_{xc(t)} - P_{x,s,c(t)}|$ can be estimated as

$$|P^\infty_{x+s,c(t)+s} - P_{x+s,c(t)+s}| \le q(a) d(x+s, c(t)+s) \le q(a) e^{-s} c_2(a, d(x, c(t))).$$

By Remark 4.1, the same estimate holds for X_∞ and c_∞.

Fix $\epsilon > 0$ and pick $R > 0$ such that $d(c(t_i), \gamma_i(c(t_i))) \le R$ for all i. Take a large enough s so that $q(a) c_2(a, R) e^{-s} < \epsilon$. Since $B(c(t_i), R+s)$ converges to $B(p, R+s)$ in $C^{1,\alpha}$-topology, and $\gamma_i \to \gamma$, we conclude that $P_{\gamma_i(c(t_i)),s,c(t_i)} \to P_{\gamma(p),s,p}$ in C^0-topology, or more formally,

$$|df_i \circ P_{\gamma_i(c(t_i)),s,c(t_i)} \circ d\gamma_i - P_{\gamma(p),s,p} \circ d\gamma| < \epsilon.$$

for large i, where f_i is the $C^{1,\alpha}$-approximation. By the estimate in the previous paragraph, $|P_{\gamma_i(c(t_i)),s,c(t_i)} - P^\infty_{\gamma_i(c(t_i))c(t_i)}| < \epsilon$ and $|P_{\gamma(p),s,p} - P^\infty_{\gamma(p),p}| < \epsilon$, so the

triangle inequality implies that $|df_i \circ P^{\infty}_{\gamma_i(c(t_i))c(t_i)} \circ d\gamma_i - P^{\infty}_{\gamma(p),p} \circ d\gamma| < 3\epsilon$ for large i. Hence $|\phi(\gamma_i) - \phi_\infty(\gamma)| < 3\epsilon$ for all large i, and as $\epsilon > 0$ is arbitrary it follows that $\phi(\gamma_i) \underset{i\to\infty}{\longrightarrow} \phi_\infty(\gamma)$. $\square$

Proposition 5.2. *Let $K = \ker \phi_\infty$ and let G_p be the isotropy subgroup of p in G. Then*
(1) $\overline{\phi(\Gamma)} = \phi_\infty(G_p) = \phi_\infty(G)$,
(2) *K acts freely on X, in particular, $K \cap G_p = \{\mathrm{id}\}$.*
(3) *The short exact sequence $1 \to K \to G \overset{\phi_\infty}{\to} \phi_\infty(G) \to 1$ splits with the splitting given by $\phi_\infty(G) \simeq G_p \hookrightarrow G$. In particular, G is a semidirect product of K and G_p.*

Proof. (1) For each $\gamma \in \Gamma$ we have $d(\gamma(c(t_i)), c(t_i)) \to 0$ as $i \to \infty$, so the constant sequence γ converges to some $g \in G_p$. By Lemma 5.1, $\phi(\gamma) \to \phi_\infty(g)$, which means $\phi(\gamma) = \phi_\infty(g)$. Thus, $\phi(\Gamma) \subset \phi_\infty(G_p)$. Now G_p is compact, so $\phi_\infty(G_p)$ is closed, and therefore, $\overline{\phi(\Gamma)} \subset \phi_\infty(G_p)$. Since $\phi_\infty(G_p) \subset \phi_\infty(G)$, it remains to show that $\phi_\infty(G) \subset \overline{\phi(\Gamma)}$. Given $\gamma \in G$, we find $\gamma_i \in \Gamma$ with $\gamma_i \to \gamma$. By Lemma 5.1, $\phi_\infty(\gamma)$ is the limit of $\phi(\gamma_i) \in \phi(\Gamma)$, so $\phi_\infty(g) \in \overline{\phi(\Gamma)}$.
(2) If $k \in K$ fixes a point x, then $1 = \phi_\infty(k) = P^{\infty}_{k(x)x} \circ dk = P^{\infty}_{xx} \circ dk = dk$. Since k is an isometry, $k = \mathrm{id}$. (3) is a formal consequence of (1) and (2). $\square$

Remark 5.3. Since G is the semidirect product of K and G_p, any $\gamma \in G$ can be uniquely written as kg with $k \in K$, $g \in G_p$, We refer to k and g respectively as the *translational part* and the *rotational part* of γ.

Remark 5.4. If Γ is discrete, then by Margulis lemma any finitely generated subgroup of Γ has a nilpotent subgroup whose index i and the degree of nilpotency d are bounded above by a constant depending only on n. (Of course, by [Bow93] Γ itself is finitely generated, as is any subgroup of Γ, but we do not need this harder fact here). The same then holds for G. Indeed, take finitely many elements g_l of G and approximate them by $\gamma_{j,l} \in \Gamma$, so they generate a finitely generated subgroup of Γ. Then $\gamma^i_{j,l}$ approximate g^i_l, and by above, g^i_l lie in a nilpotent subgroup of Γ. Hence a d-fold iterated commutator in $g^i_{j,l}$'s is trivial for all j, and then so is the corresponding commutator in g^i_l's. Hence G is nilpotent by [Rag72, Lemma VIII.8.17]

6. Controlling injectivity radius

We continue working with notations of Section 5, except now we also assume that Γ is discrete. The family $(X, c(t), \Gamma)$ may have many converging subsequences with limits of the form (X_∞, p, G). We denote by $K(p)$ the K-orbit of p, where K is the kernel of the rotation homomorphism $G \to O(n)$. The

goal of this section is to find a common lower bound, on the normal injectivity radii of $K(p)$'s.

Proposition 6.1. *There exists a constant $f(a)$ such that for each $x \in K(p)$, the norm of the second fundamental form II_x of $K(p)$ at x is bounded above by $f(a)$.*

Proof. Since K acts by isometries $|II_x| = |II_p|$ for any $x \in K(p)$ so we can assume $x = p$. Let $X, Y \in T_pK(p)$ be unit tangent vectors. Extend Y to a left invariant vector field on $K(p)$, and let $\alpha(t) = exp(tX)(p)$ be the orbit of p under the one-parameter subgroup generated by X. Since $II_p(X,Y)$ is the normal component of $\nabla_X Y(p)$, it suffices to show that $|\nabla_X Y| \le f(a)$. Let $P^{\alpha}_{p,\alpha(t)}$ be the parallel transport from p to $\alpha(t)$ along α. By Section 4 $|P_{p,\alpha(t)} - P^{\infty}_{p,\alpha(t)}| \le q(a)d(p,\alpha(t)) \le 2q(a)t$ for all small t.

A similar argument shows that $|P_{p,\alpha(t)} - P^{\alpha}_{p,\alpha(t)}| \le 2q(a)t$ for all small t. Indeed, look at the "ruled" surface obtained by joining p to the points of α near p. If we approximate α by a piecewise geodesic curve $p\alpha(t_1)\cdots\alpha(t_k)$, where $\alpha(t_k) = q$ is some fixed point near p, then the area of the surface can be computed as the limit as $k \to \infty$ of the sum of the areas of geodesic triangles $p\alpha(t_i)\alpha(t_{i+1})$. The area of each triangle is bounded above by $d(\alpha(t_i)\alpha(t_{i+1}))$, so the area of the ruled surface is bounded above the length of α from p to q, which is at most $2t$, for small t.

Therefore, $|P^{\infty}_{p,\alpha(t)} - P^{\alpha}_{p,\alpha(t)}| \le 4q(a)t = f(a)t$ by the triangle inequality, so $|P^{\infty}_{\alpha(t),p}Y - P^{\alpha}_{\alpha(t),p}Y| \le f(a)t$. On the other hand, $P^{\infty}_{\alpha(t),p}Y = Y(p)$ because Y is left-invariant, and since elements of K have trivial rotational parts. Thus $|P^{\alpha}_{\alpha(t),p}Y - Y(p)| \le f(a)t$, which by definition of covariant derivative implies that $|\nabla_X Y(p)| \le f(a)$. □

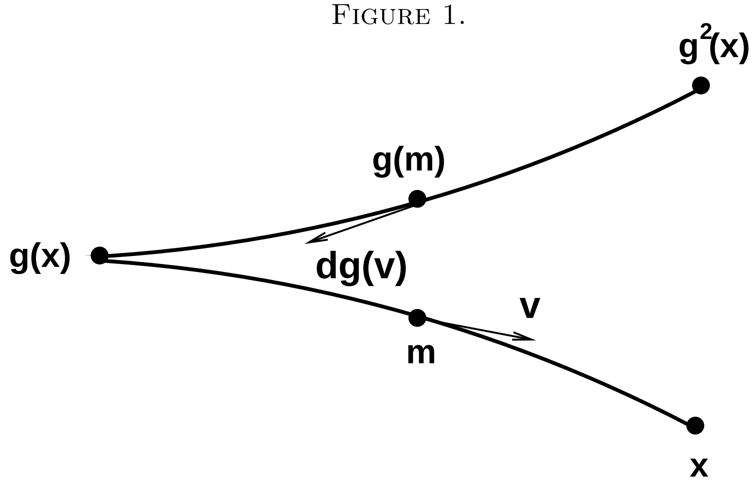
Corollary 6.2. (i) *There exists $r(a) > 0$ such that if C is the connected component of $K(p) \bigcap B_{r(a)}(p)$ that contains p, and if $x \in B_{r(a)}(p)$ is the endpoint of the geodesic segment $[x,p]$ that is perpendicular to C at p, then $d(x,c) > d(x,p)$ for any $c \in C \setminus \{p\}$.*
(ii) *If there exists $s < r(a)$ such that $K(p) \bigcap B_s(p)$ is connected, then the normal injectivity radius of $K(p)$ is $\ge s/3$.*

Proof. (i) The metric on X_∞ can be approximated in C^1-topology by smooth metrics with almost the same two-sided negative curvature bounds and infinite injectivity radius [Nik89]. Also C^1-closeness of metrics implies C^0-closeness of Levi-Civita connections, and hence almost the same bounds on the second fundamental forms of C. Now for the smooth metrics as above the assertion of (i) is well-known, and after choosing a slightly smaller $r(a)$, it passes to the limits so we also get it for X_∞.

(ii) Consider two arbitrary geodesic segments of equal length $\leq s/3$ that start at $K(p)$, are normal to $K(p)$ and have the same endpoint. Since K acts isometrically on X_∞ and transitively on $K(p)$, we can assume that one of the segments starts at p. By the triangle inequality, the other segment starts at a point of C, so by part (i) the segments have to coincide. □

Remark 6.3. The proof that $K(p)\bigcap B_s(p)$ is connected for some $s < r(a)$ independent of the converging subsequence $(X, c(t_i), \Gamma)$ occupies the rest of this section, and this is the only place in the paper that uses Bowditch's theorem [Bow93] that Γ is finitely generated. Other key ingredients are the existence of approximate square roots in finitely generated nilpotent groups (see Appendix A), and the following comparison lemma that relates the displacement of an element of Γ to the displacement of its square root.

FIGURE 1.



Lemma 6.4. *Let U be the neighborhood of $1 \in O(n)$ that consists of all $A \in O(n)$ satisfying $|Av - v| < 1$ for any unit vector in $v \in \mathbb{R}^n$. There exists a function $f\colon (0,\infty) \to (0,\infty)$ such that $f(r) \to 0$ as $r \to 0$ and $d(g(x), x) \leq f(d(g^2(x), x))$, for any $x \in X$ and any $g \in \Gamma$ with $\phi(g) \in U$.*

Proof. Define $f(r)$ to be the supremum of numbers $d(g(x), x)$ over all $x \in X$ and $g \in \Gamma$ with $\phi(g) \in U$ satisfying $r = d(x, g^2(x))$.

To see that $f(r) < \infty$ take an arbitrary $x \in X$, $g \in \Gamma$ with $r = d(x, g^2(x))$ and let $R = d(x, g(x))$. Look at the geodesic triangle in X with vertices x, $g(x)$, $g^2(x)$ (see Figure 1). Arguing by contradiction, assume that by choosing g, x one can make R arbitrary large while keeping r fixed. The geodesic triangle then becomes very long and thin. Let m be the midpoint of the geodesic segment $[x, g(x)]$, so that $g(m)$ is the midpoint of the geodesic segment

$[g(x), g^2(x)]$. By exponential convergence of geodesics and comparison with the hyperbolic plane of sec $= -1$, we get $d(m, g(m)) \leq C(r)e^{-R/2}$, which is small since R is large. So $P^{\infty}_{g(m)m}$ is close to $P_{g(m)m}$, which in turn is close to $P_{g(x)m} \circ P_{g(m)g(x)}$, since the geodesic triangle with vertices m, $g(m)$, $g(x)$ has small area. Thus, $\phi(g)$ is close to $P_{g(x)m} \circ P_{g(m)g(x)} \circ dg$. Let v be the unit vector tangent to $[x, g(x)]$ at m and pointing towards x. Then $dg(v)$ is tangent to $[g(x), g^2(x)]$ at $g(m)$ and is pointing towards $g(x)$. Since the geodesic triangle with vertices x, $g(x)$, $g^2(x)$ has small angle at $g(x)$, the map $P_{g(x)m} \circ P_{g(m)g(x)}$ takes $dg(v)$ to a vector that is close to $-v$. This gives a contradiction since $|\phi(g)(v) - v| \leq 1$.

A similar argument yields that $f(r) \to 0$ as $r \to 0$. Namely, if one can make $d(g^2(x), x)$ arbitrary small while keeping $d(g(x), x)$ bounded below, then the geodesic triangle with vertices x, $g(x)$, $g^2(x)$ becomes thin, and we get a contradiction exactly as above. $\square$

Proposition 6.5. *Let $r(a)$ be the constant of Corollary 6.2. Then there exists a positive $s \leq r(a)$, depending only on X, c, Γ, such that for any converging sequence $(X, c(t_i), \Gamma) \to (X_\infty, p, G)$, the normal injectivity radius of $K(p)$ is $\geq s$.*

Proof. By Corollary 6.2, it suffices to find a universal s such that $K^s := K(p) \bigcap B_s(p)$ is connected. Let K^s_0 be the component of K^s containing p.

By [Bow93] Γ is finitely generated, hence by Margulis lemma [BGS85], Γ contains a normal nilpotent subgroup $\tilde{\Gamma}$ of index $i \leq i(a, n)$. Therefore $H = \overline{\phi(\Gamma)}$ is virtually nilpotent. Hence, its identity component H_0 is abelian, since compact connected nilpotent Lie groups are abelian. Then $\phi^{-1}(H_0)$ is a subgroup of finite index in Γ. Let $\Gamma' = \tilde{\Gamma} \cap \phi^{-1}(H_0)$. Clearly, $[\Gamma : \Gamma'] = k = k(\phi)$ is also finite.

We first give a proof in case $\Gamma = \Gamma'$. Arguing by contradiction, suppose that for any $s > 0$ there exists a sequence $(X, c(t_i), \Gamma) \to (X_\infty, p, G)$ and a point $\gamma(p) \in K^s \setminus K^s_0$ with $d(p, \gamma(p)) < s$. By possibly making $d(p, \gamma(p))$ smaller, we can choose $\gamma(p)$ so that $d(p, \gamma(p))$ is the distance from p to $K^s \setminus K^s_0$. By the first variation formula the geodesic segment $[p, \gamma(p)]$ is perpendicular to K^s_0. The next goal is to construct the square root of γ with no rotational part and displacement bounded by $f(d(p, \gamma(p)))$, where f is the function of Lemma 6.4.

Take $\gamma_i \in \Gamma$ that converge to γ. Since Γ is finitely generated, we can apply Lemma A.1 to find an (independent of i) finite set $F \subset \Gamma$ such that each γ_i can be written as $\gamma_i = g_i^2 f_i$ with $f_i \in F$. We can further write each g_i as the product $g_i = x_i r_i$, where x_i has a small rotational part and r_i is close to a rotation, namely we let r_i be an element of Γ that is close to $\phi(g_i)$ and let

$x_i = g_i r_i^{-1}$. Thus $\gamma_i = (x_i r_i)^2 f_i$ and

$$\gamma_i = x_i r_i x_i r_i f_i = x_i^2 [x_i^{-1} r_i] r_i^2 f_i$$

Applying Lemma A.2 to $x_i^2[x_i^{-1}r_i]$ we see that γ_i can be written as $(x_i h_i)^2 f_i' r_i^2 f_i$ with $h_i \in [\Gamma, \Gamma]$, $f_i' \in F'$. Since $\phi(\Gamma)$ ia abelian, we have $\phi(h_i) = 1$ and hence $\phi(x_i h_i) = \phi(x_i)$ is small. Since F, F' are independent of i, each element of F, F' is close to a rotation for large i. So $f_i' r_i^2 f_i$ is close to a rotation and since both $\phi(\gamma_i)$ and $\phi(x_i h_i)$ are small, $f_i' r_i^2 f_i$ subconverges to the identity. Thus, $(x_i h_i)^2$ subconverges to γ, and so we might as well assume that $\gamma_i = (x_i h_i)^2$ in the beginning. By Lemma 6.4,

$$d(c(t_i), x_i h_i(c(t_i))) \leq f(d(c(t_i), (x_i h_i)^2(c(t_i)))$$

where the right hand converges to $f(d(p, \gamma(p)))$. Hence $x_i h_i$ subconverges to $w \in K$ such that $w^2 = \gamma$ and $d(p, w(p)) \leq f(d(p, \gamma(p)))$.

Since w has no rotational part, $P^{\infty}_{pw(p)} = dw_p$. By assumption s can be taken arbitrary small, so we can assume that $f(d(p, \gamma(p)))$ is small, in particular, $w(p) \in K^s$. So $P_{pw(p)}$ is close to dw_p. Hence if v is the unit vector tangent to $[p, w(p)]$ at p and pointing towards $w(p)$, then $P_{pw(p)}(v)$ is close to $dw_p(v)$. Therefore, $w(p)$ is close to the midpoint of $[p, w^2(p)] = [p, \gamma(p)]$. Since $[p, \gamma(p)]$ is perpendicular to K_0^s and $d(p, \gamma(p)) < r(a)$, it is clear that $w(p) \notin K_0^s$. This contradicts the minimality of $d(p, \gamma(p))$ and completes the proof in case $\Gamma = \Gamma'$.

We now turn to the general case. Let G' be the subset of G that consists of limits of elements of Γ' under the convergence $(X, c(t_i), \Gamma) \to (X_\infty, p, G)$. It is straightforward to check that G' is a closed subgroup of G of index $\leq k$. Thus the limit of any converging subsequence of $(X, c(t_i), \Gamma')$ has to equal to (X_∞, p, G'), therefore in fact, $(X, c(t_i), \Gamma')$ converges to (X_∞, p, G'). Since the rotation homomorphism of G restricts to the rotation homomorphism of G', the translational part K' of G' is $G' \cap K$. In particular, $|K : K'| \leq k$ hence the identity components of K and K' coincide. Using the first part of the proof, we fix s such that $K'(p) \bigcap B_s(p)$ is connected. Thus $K'(p) \bigcap B_s(p) = K_0^s$.

Now let $\gamma(p) \in K^s \setminus K_0^s$ such that $d(p, \gamma(p))$ is the distance from p to $K^s \setminus K_0^s$. Then the geodesic segment $[p, \gamma(p)]$ is perpendicular to K_0^s by the first variation formula. Arguing by contradiction suppose that $d(p, \gamma(p))$ can be arbitrary small. Then by triangle inequality $\gamma^j(p)$ is close to p for $j = 1, \ldots, k$. Also $\gamma^k \in K'$, so in fact $\gamma^k(p) \in K_0^s$ because $K'(p) \bigcap B_s(p) = K_0^s$.

On the other hand, since γ^j's have no rotational part, the argument used above to prove that $w(p)$ is close to the midpoint of $[p, w^2(p)]$ shows that the points $\gamma^j(p)$ almost lie on a geodesic segment $[p, \gamma^k(p)]$. Then the segments $[p, \gamma^k(p)]$ and $[p, \gamma(p)]$ have almost the same direction, so $[p, \gamma^k(p)]$ is almost perpendicular to K_0^s. Hence by Corollary 6.2, if $d(p, \gamma(p))$ is small enough, then $\gamma^k(p) \notin K_0^s$ which is a contradiction. $\square$

Remark 6.6. Although it is not needed for the proof of Theorem 1.1, note that all possible limits of $(X/\Gamma, \sigma(t_i))$ have the same dimension independent of the sequence $t_i \to \infty$. Consider all possible limits with fixed $\dim(K)$, and look at the space X_∞/K. It has a lower bound on the injectivity radius for points near the projection $\bar{p}$ of p, and hence $\mathrm{vol}(B(\bar{p}, 1)) > c > 0$ in all such spaces, where c is independent of the converging sequence. By Proposition 5.2, the isotropy group G_p is the same for all possible limits and moreover, the G_p-actions on T_pX_∞ are all equivalent. Also by Lemma A.3 the identity component G_p^{id} of G_p commutes with the identity component of K, hence G_p^{id} fixes pointwise the component of $K(p)$ containing p. Thus the G_p^{id}-actions on $T_{\bar{p}}X_\infty/K$ are all equivalent. This implies that a unit ball in X_∞/G has volume $> c' > 0$ with c' only depending on $\dim(K)$. Hence all limits of the same dimension form a closed subset among all limits, therefore, the space of all limits is the union of these closed sets. On the other hand, the space of all limits is connected by Lemma 6.7 below, thus all the limits have the same dimension.

Lemma 6.7. *If $\gamma\colon [0,\infty) \to Z$ is a continuous precompact curve in a metric space Z, then the space $Lim(\gamma)$ of all possible subsequential limits $\lim_{t_i\to\infty}\gamma(t_i)$ is connected.*

Proof. If $Lim(\gamma)$ is not connected, then we can write it as a disjoint union of closed (and hence compact) sets $Lim(\gamma) = A \sqcup B$. Then $U_\epsilon(A) \cap U_\epsilon(B) = \emptyset$ for some $\epsilon > 0$, where $U_\epsilon(S)$ denotes the ϵ-neighborhood of S. Let $\gamma(t_i) \to a \in A$ and $\gamma(t_i') \to b \in B$. Arguing by contradiction, we see that the curve $\gamma|_{[t_i,t_i']}$ lies in $U_\epsilon(Lim(\gamma)) = U_\epsilon(A) \cup U_\epsilon(B)$ for all large i. Clearly, $\gamma(t_i) \in U_\epsilon(A)$ and $\gamma(t_i') \in U_\epsilon(B)$ for all large i, which contradicts $U_\epsilon(A) \cap U_\epsilon(B) = \emptyset$. $\square$

7. Product structure at infinity

In the next two sections we apply the critical point theory for distance functions to show the following.

Theorem 7.1. *For each large t, the horosphere quotient H_t/Γ is diffeomorphic to the normal bundle of an orbit of an N-structure on H_t/Γ.*

Proof. Let $\sigma(t)$ be the projection of $c(t)$ to X/Γ. Given a converging sequence $(X, c(t_i), \Gamma) \to (X_\infty, p, G)$, the sequence of pointed Riemannian manifolds $(X/\Gamma, \sigma(t_i))$ converges in the pointed Gromov-Hausdorff topology to a pointed Alexandrov space $(Y, q) := (X_\infty/G, q)$ with curvature bounded below by $-a^2$.

The identity component K_{id} of K is normal in K, hence its p-orbit $K_{\mathrm{id}}(p)$ is invariant under the action of G_p. Let $r \ll s$ be a positive constant to be determined later, where s comes from Proposition 6.5. The $3r$-tubular

neighborhood of $K_{\rm id}(p)$ is also G_p-invariant, so the ball $B_{3r}(q)$ is isometric to the G-quotient of this tubular neighborhood. By Proposition 6.5, any $x \in B_{3r}(q)$ can be joined to q by a unique shortest geodesic segment $[q,x] \subset B_{3r}(q)$.

Recall that in general a distance function $d(\cdot,q)$ on an Alexandrov space is called *regular at the point* x if there exist a segment emanating from x that forms the angle $> \pi/2$ with any shortest segment joining x to q.

In our case the function $d(\cdot,q)$ is regular at any $x \in B_{3r}(q) \setminus \{q\}$.

Let $w(a) > 1$ be a constant depending only on a that will be specified later. By angle comparison the function $d(\cdot,\sigma(t))$ on

$$A_r(\sigma(t)) = \{x \in B_s(\sigma(t))\colon\ d(x,\sigma(t)) \in [r/w(a), w(a)r]\}$$

is regular provided the Gromov-Hausdorff distance between $B_s(q)$ and $B_s(\sigma(t))$ is $\ll r/w(a)$. Because the family $\{B_s(\sigma(t))\}$ is precompact in the Gromov-Hausdorff topology, Proposition 6.5 implies that the function $d(\cdot,q)$ is regular on $A_r(\sigma(t))$ for all $t \geq t_0$ with sufficiently large t_0.

We denote by H_t the horosphere centered at $c(\infty)$ that contains $c(t)$. Since the second fundamental form of H_t, is bounded in terms of a, any short segment joining nearby points of H_t/Γ is almost tangent to H_t/Γ. Hence by taking r sufficiently small, we can assume that for all $t \geq t_0$ and each $x \in A_r(\sigma(t))$ there exists a unit vector $\lambda \in T_x(H_t/\Gamma)$ that forms the angle $\alpha_{\lambda,[x,\sigma(t)]} \in [\frac{2\pi}{3}, \pi]$ with any shortest segment $[x,\sigma(t)]$. By the first variation formula, the derivative of $d(\cdot,q)$ in the direction of λ equals to the minimum of $-\cos\alpha_{\lambda,[x,\sigma(t)]}$ over all shortest segments $[x,\sigma(t)]$ and by above it lies in $[\frac{1}{2},1]$.

The distance function on X/Γ need not be smooth, and for what follows it is convenient to replace $d(\cdot,\sigma(t))$ by its average over a small ball $B_\delta(\sigma(t))$ as follows. Given $\delta \ll r$, define $f\colon\ X/\Gamma \to \mathbb{R}$ by

$$f(x) = \frac{1}{\mathrm{vol} B_\delta(\sigma(t))} \int_{B_\delta(\sigma(t))} d(x,y)dy$$

where $x \in H_t/\Gamma$ (i.e. $t = b(x)$). Now f is C^1 1-Lipschitz function with $|f(x) - d(x,\sigma(t))| \leq \delta$ for any $x \in A_r(\sigma(t))$. Observe that for any $\eta \in T_x(X/\Gamma)$

$$df_x(\eta) = \frac{1}{\mathrm{vol} B_\delta(\sigma(t))} \int_{B_\delta(\sigma(t))} (-\cos\alpha_{\eta,[x,y]})dy \tag{7.2}$$

Also note that since $\delta \ll r$ and $\sec(X/\Gamma) \geq -a^2$, for all large t if $x \in A_r(\sigma(t))$ and $y \in B_\delta(\sigma(t))$ there is a point z such that $d(z,x) \approx d(x,y)$ and $d(z,y) \approx 2d(x,y)$. Therefore, the angle corresponding to x in the comparison triangle in the space of $\sec \equiv -a^2$ is almost π. By Toponogov comparison, the angle at x in any geodesic triangle Δxyz is almost π. By (7.2), this implies that if η

is a direction of any shortest segment connecting x to z, then $df_x(\eta) \in [\frac{1}{2}, 1]$ provided δ is small enough.

By gluing λ's via a partition of unity, we obtain a C^1 unit vector field Λ that is tangent to H_t/Γ and defined for all $t \geq t_0$ and $x \in A_r(\sigma(t))$, and such that $df_x(\Lambda) \in [\frac{1}{4}, 1]$ if δ is sufficiently small. Then $S_r = \{x \in X/\Gamma : f(x) = r\}$ is a properly embedded C^1-hypersurface in X/Γ that is transverse to Λ. Also the compact submanifold $S_t(t) := S_r \bigcap H_t/\Gamma$ is δ-close to the metric r-sphere in H_t/Γ centered at $\sigma(t)$. Furthermore, $A(r,t) := A_r(\sigma(t)) \cap H_t/\Gamma$ is C^1-diffeomorphic to the product $S_r(t) \times [r/w(a), w(a)r]$.

Here we are only interested in the part of X/Γ with $t \geq t_0$. There the Busemann function $X/\Gamma \to \mathbb{R}$ restricts to a C^1-submersion $S_r \to [t_0, \infty)$, because otherwise at some point the tangent spaces of S_r and H_t/Γ coincide by dimension reasons, so that S_r cannot be transverse to $\Lambda \in T(H_t/\Gamma)$. By construction the submersion is proper, hence it is a C^1-fiber bundle, which is C^1-trivial by the covering homotopy theorem. The trivialization defines a C^1-isotopy $F\colon S_r(t_0) \times [t_0, \infty) \to X/\Gamma$ such that $S_r(t_0) \times \{t\}$ is mapped onto $S_r(t)$.

We push this isotopy along the Busemann flow back into H_{t_0}/Γ by setting $G(t,x) = b_{t_0-t}(F(x,t))$ for any $t \geq t_0$, and $x \in S_r(t_0)$, to get the C^1-isotopy $G\colon S_r(t_0) \times [t_0, \infty) \to H_{t_0}/\Gamma$.

The Busemann flow induces a C^1-diffeomorphism $H_t/\Gamma \to H_{t_0}/\Gamma$, so around each submanifold $b_{t_0-t}(S_r(t))$ there is a "tubular neighborhood" $b_{t_0-t}(A(r,t))$.

By the exponential convergence of geodesics, one can choose $w(a)$ in the definition of $A_r(\sigma(t))$ so that for any $t \geq t_0$ there exists $t' \geq t+1$ such that $b_{t_0-t'}(S_r(t'))$ is contained in $b_{t_0-t}(A(r,t))$ and is disjoint from $b_{t_0-t}(S_r(t))$. By the following elementary lemma, the region between $b_{t_0-t}(S_r(t))$ and $b_{t_0-t'}(S_r(t'))$ is C^1-diffeomorphic to $S_r(t) \times [0,1]$.

Lemma 7.3. *Let M a closed smooth manifold M and $F_t\colon M \to M \times \mathbb{R}$ be a C^1-isotopy with $F_0(M) = M \times \{0\}$. If $F_0(M)$ and $F_s(M)$ are disjoint for some s, then the region between $F_0(M)$ and $F_s(M)$ is diffeomorphic to $M \times [0,1]$.*

Proof. By the isotopy extension theorem [Cer61, p 293] we can extend the isotopy F_t to an ambient C^1-isotopy which is identity outside a compact subset of $M \times \mathbb{R}$. Assume without loss of generality that $s < 0$, and then take $n > 0$ so large that the isotopy is identity on $M \times \{n\}$. Then by restricting the ambient isotopy to the region between $M \times \{n\}$, $M \times \{0\}$, we get a diffeomorphism of the region between the region between $M \times \{n\}$, $M \times \{0\}$ onto the region between $M \times \{n\}$, $F_s(M)$. The former region is the product, so is the latter. But the latter region is diffeomorphic to the region between $M \times \{0\}$, $F_s(M)$, because the region between $M \times \{n\}$, $M \times \{0\}$ is $M \times [0,1]$. □

By gluing a countable number of such diffeomorphisms together we conclude that for all sufficiently large t

$$(7.4) \qquad \begin{array}{c}(H_t/\Gamma)\setminus U(r,t) \text{ is } C^1 \text{ diffeomorphic to } [t,\infty)\times S_r(t)\\ \text{where } U(r,t)=\{x\in H_t/\Gamma\colon\ f(x)<r\}\end{array}$$

8. Tubular neighborhood of an orbit

It remains to understand the topology of $U(r,t)$, and we do so for large enough t and small enough r. The proof involves the collapsing theory developed in [CFG92] and the geometry of Alexandrov spaces (for which we refer to [BGP92] and Appendix D).

Let us look at a converging sequence $(H_{t_i}/\Gamma, \sigma(t_i))\to(H,q)$. First, we replace the metric on H_{t_i}/Γ with an invariant Riemannian metric which is ϵ_i close to H_{t_i}/Γ in C^1 topology and is $A(\epsilon_i)$-regular [Shi89, CFG92, Nik89], where $\epsilon_i\to 0$ as $i\to\infty$. Also, spaces H_{t_i}/Γ with the new metrics have uniform curvature bound $|\sec|\le C'=C'$ [Shi89] that depends only on the original curvature bound of H_{t_i}/Γ. The collapsing theory [CFG92] yields, for each i, the following commutative diagram given by the invariant metric h_i on H_{t_i}/Γ.

$$\begin{array}{ccc} FB_i & \xrightarrow{\eta_i} & Y_i\\ \downarrow & & \downarrow\\ B_i & \xrightarrow{\bar\eta_i} & X_i\end{array}$$

Here B_i is the ball $B(\sigma(t_i),1)$ in the metric h_i, and FB_i is the frame bundle of B_i. The vertical arrows are quotient maps under isometric $O(n)$-actions and η_i is a Riemannian submersion given by the N-structure on FB_i. Clearly the induced map $\bar\eta_i$ is a submetry (see Appendix D for background on submetries). By Lemma D.3, the Toponogov comparison with curv $\ge -C'$ holds for any triangle with vertices in $B(\bar\eta_i(\sigma(t_i)),\frac14)$ for all large i.

Since we will only be interested in the geometry of X_i inside $\frac18$-neighborhood of $\bar\eta_i(\sigma(t_i))$, we will treat X_i's as Alexandrov spaces.

Note that $X_i\overset{G-H}{\longrightarrow}\bar X=B(q,1)$ and $\dim X_i=\dim\bar X$ for all large i. We claim that there exists an $R>0$ and a sequence $q_i\in X_i$ converging to q such that $d(\cdot,q_i)$ has no critical points in $B(q_i,R)$ for all large i. Consider two cases depending on whether q lies on the boundary of the Alexandrov space $\bar X$:

Case 1: Suppose $q\notin\partial\bar X$.

Since $\bar X$ has curv $\ge -C'$ in comparison sense, by [Kap02], the exists a strictly concave function u on a $B(q,R)$ for some $R\ll 1$ such that it has a maximum at q and the superlevel sets are compact. By possibly making R smaller we

can assume that $R < d(q, \partial\bar{X})$. This function is constructed by taking averages and minima of distance functions. Therefore, it naturally lifts to a function u_i on X_i such that u_i converges uniformly to u. By [Kap02, Lemma 4.2], the lifts u_i are strictly concave on $B(\bar{\eta}_i(\sigma(t_i)), R/2)$ for all large i. Let q_i be the point of maximim of u_i. By uniqueness of the maximim $q_i \underset{i\to\infty}{\longrightarrow} q$. By Lemma D.1, $d(\cdot, q_i)$ has no critical points in $B(q_i, R/3)$ for all large i.

Case 2: Suppose now that $q \in \partial\bar{X}$. Denote by $D\bar{X}$ and DX_i the doubles of $\bar{X}$ and X_i along the boundary and let ι be the canonical involution. By [Per91], the doubles are also Alexandrov spaces with curv $\geq -C'$. It is clear that $DX_i \overset{G-H}{\longrightarrow} D\bar{X}$. By construction, we can chose u and u_i to be ι-invariant. As before, let q_i be the point of maximum of u_i. Since $u_i(q_i) = u_i(\iota(q_i))$, by uniqueness of maximums of strictly concave functions we see that q_i must lie on ∂X_i. Again by Lemma D.1, $d(\cdot, q_i)$ has no critical points in $B(q_i, R/3)$ in DX_i for all large i and hence the same is true for $d(\cdot, q_i)$ in $B(q_i, R/3)$.

This immediately implies that the distance function $d(\cdot, O_i)$ to the orbit O_i over q_i has no critical points in the $R/3$ neighborhood $U_{R/3}(O_i)$ of O_i for all large i. Indeed, let $x \in U_{R/3}(O_i)\backslash O_i$ and let $\gamma(t)$ be a geodesic starting at $\bar{\eta}_i(x)$ such that $\frac{d}{dt}d(\gamma(\cdot), q_i)'|_{t=0} > 0$. Since $\bar{\eta}_i\colon (H(t_i)/\Gamma, h_i) \to X_i$ is a submetry, there exists $\tilde{\gamma}$, a horizontal lift of γ starting at x. Then $d(\tilde{\gamma}(t), O_i) = d(\gamma(t), q_i)$ and hence $d(\tilde{\gamma}(t), O_i)'|_{t=0} = d(\gamma(t), q_i)'|_{t=0} > 0$.

Therefore $U_r(O_i)$ is diffeomorphic to the total space of the normal bundle to O_i in H_{t_i}/Γ for any $r \leq R/3$. Since O_i is Hausdorff close to $\sigma(t_i)$, the same is true for $U(r, t_i)$.

Combining this with (7.4), we conclude that H_{t_i}/Γ is diffeomorphic to the total space of the normal bundle to O_i in H_{t_i}/Γ for all sufficiently large i. Finally, since the above proof works for any sequence $t_i \to \infty$, arguing by contradiction we conclude that H_t/Γ is diffeomorphic to the normal bundle of an orbit of an N-structure for all sufficiently large t. This completes the proof of Theorem 7.1. ☐

Remark 8.1. The reader may be wondering why we work with the Alexandrov spaces X_i instead of the Riemannian manifolds Y_i. This is because the curvature of Y_i may tend to $\pm\infty$ as $i \to \infty$, which makes it hard to control the geometry of Y_i's. If instead of $\epsilon_i \to 0$, we take ϵ_i equal to a small positive constant ϵ, then $|\sec(Y_i)| \leq C(\epsilon)$, but then it may happen that the injectivity radius of the GH-limit of Y_i's is $\ll \epsilon$, so we cannot translate the lower bound on the injectivity radius from Y_i to H_{t_i}/Γ.

Remark 8.2. By Theorem 7.1 each orbit O_{q_i} as above is homotopy equivalent to X/Γ. Thus all O_{q_i}'s are homotopy equivalent, hence they are all affine diffeomorphic (see e.g. [Wil00b, Theorem 2]).

9. Normal bundle is flat

Theorem 9.1. *For each large t, the horosphere quotient H_t/Γ admits an N-structure that has an orbit O_t such that the normal bundle to O_t is a flat Euclidean vector bundle with total space diffeomorphic to H_t/Γ.*

Arguing by contradiction, it suffices to prove the theorem for any sequence $t_i \to \infty$ such that H_{t_i}/Γ converges in pointed Gromov-Hausdorff topology. We fix such a sequence and assume for the rest of the proof that t belongs to the sequence.

We denote by g_t the C^1-Riemannian metric on the horosphere quotient H_t/Γ induced by the ambient metric (M, g). Fix a small positive $\epsilon > 0$ to be determined later; this constant will only depend on (M, g). By Theorem 7.1 and [CFG92, Nik89], for all large t there exists an N-structure on H_t/Γ with an orbit O_t such that the normal bundle to O_t is diffeomorphic to H_t/Γ. Also all orbits of the N-structure have diameter $< \epsilon$ with respect to an invariant metric h_t that is ϵ-close to g_t in uniform C^1-topology, i.e. $|g_t - h_t| < \epsilon$, $|\nabla_{g_t} - \nabla_{h_t}| < \epsilon$. It remains to show that for all large t, the normal bundle to O_t in H_t/Γ is flat Euclidean. The proof break into two independent parts.

In Section 9.1 we find a stratum F_t of the N-structure on H_t/Γ such that F_t is an h_t-totally-geodesic closed submanifold that contains O_t with flat normal bundle. This uses only general properties of N-structures.

In Section 9.2 we show that the restriction to O_t of the normal bundle of F_t in H_t/Γ is flat, for all large t. This uses the flat connection of Section 4 and the fact that $O_t \to H_t/\Gamma$ is a homotopy equivalence.

9.1. Normal bundle in a stratum is flat.

Throughout Section 9.1 we suppress the index t, and write O in place of O_t, etc. Let V be the tubular neighborhood of O that is sufficiently small so that all orbits in V have dimension $\geq \dim(O)$. Let $\tilde{O}$, $\tilde{V}$ be their universal covers. According to [CFG92, pp. 364–365], the group $\mathrm{Iso}(\tilde{V})$ contains a connected (but not necessarily simply-connected) nilpotent subgroup N that stabilizes $\tilde{O}$, acts transitively on $\tilde{O}$, and also $\Lambda = N \cap \pi_1(V)$ is a finite index subgroup of $\pi_1(V)$ and is a lattice in N. The following lemma is implicit in [CFG92].

Lemma 9.2. *The subgroup H of $\mathrm{Iso}(\tilde{V})$ generated by N and $\pi_1(V)$ is closed, N is the identity component in H and the index of N in H is finite.*

Proof. Λ is a cocompact discrete subgroup of N and also of its closure $\bar{N}$ in $\mathrm{Iso}(\tilde{V})$. Since $\dim(N)$, $\dim(\bar{N})$ are both equal to the cohomological dimension of Λ, we get $N = \bar{N}$. Now let Λ_0 be a maximal finite index *normal* subgroup of $\pi_1(\tilde{V})$ that is contained in Λ. If $\gamma \in \pi_1(V)$, then $N \cap \gamma N \gamma^{-1}$ contains Λ_0

as a cocompact discrete subgroup, so as before $\dim(N)$, $\dim(N \cap \gamma N \gamma^{-1})$ are both equal to the cohomological dimension of Λ_0, so $N = N \cap \gamma N \gamma^{-1}$, and N is normalized by $\pi_1(V)$. Thus N is normal in H, and $\Lambda = \Lambda_0$. Since N is connected, it remains to show that $|H : N|$ is finite. Since N and $\pi_1(V)$ generate H, the finite subgroup $\pi_1(V)/\Lambda$ of H/N generates H/N, hence $\pi_1(V)/\Lambda = H/N$. □

Let I be the intersection of the isotropy subgroups of H of the points of $\tilde{O}$. Since $\tilde{O}$ is H-invariant, I is normal in H. The fixed point set of I is a totally geodesic submanifold $\tilde{F}$ of $\tilde{V}$. The H-action on $\tilde{F}$ descends to an H/I-action on $\tilde{F}$. Since $\pi_1(V)$ is torsion free and discrete, $\pi_1(V) \cap I$ is trivial, and we identify $\pi_1(V)$ with its image in H/I. Denote the projection of $\tilde{F}$ into V by F.

Lemma 9.3. *The normal bundle to O in F is flat.*

Proof. The group N acts transitively on $\tilde{O}$ so all isotropy subgroups for the N-action on $\tilde{O}$ are conjugate. Since they are also compact, they lie in the center of N [Iwa45], in particular all the isotropy subgroups are equal, and hence each of them is equal to $I \cap N$. In particular, $N/(I \cap N)$ acts freely and transitively on $\tilde{O}$. Since $\tilde{O}$ is simply-connected, so is $N/(I \cap N)$. Thus, $I \cap N$ is the maximal compact subgroup of N, hence by Lemma A.3 $I \cap N$ is a torus, which we denote T. The torus is the identity component of the compact group I, because $|I : I \cap N| \le |H : N| < \infty$. Since N/T acts freely and transitively on $\tilde{O}$, we can choose a trivialization of ν, the normal bundle of F in $\tilde{V}$ that is invariant under the left translations by N/T. Namely, let $e \in \tilde{O}$ be the point corresponding to $1 \in N/T$ under the diffeomorphism $N/T \cong \tilde{O}$. Fix an isomorphism $\phi\colon \nu_e \to \{e\} \times \mathbb{R}^k$, and then extend it to the N/T-left-invariant isomorphism $\nu \cong \tilde{O} \times \mathbb{R}^k$. Now take $\gamma \in \pi_1(V)$, and $x \in \tilde{O}$. Using the above trivialization we define the *rotational part* of γ as the automorphism of $\{e\} \times \mathbb{R}^k$ given by

$$\phi \circ dL_{\gamma(x)^{-1}} \circ d\gamma \circ dL_x \circ \phi^{-1},$$

where dL_x is the differential of the left translation by $x \in N/T$. Since O is an infranilmanifold, $\pi_1(V)$ acts on $\tilde{O}$ by affine transformations, that is if $y \in \tilde{O}$, then $\gamma(y) = n_\gamma \cdot A_\gamma(y)$, where $n_\gamma \in N/T$ and A_γ is a Lie group automorphism of N/T. Hence, for $z \in \tilde{O}$, we get

$$(L_{\gamma(x)^{-1}} \circ \gamma \circ L_x)(z) = \gamma(x)^{-1} \cdot \gamma(xz) = A_\gamma(x)^{-1} \cdot n_\gamma^{-1} \cdot n_\gamma \cdot A_\gamma(xz) = A_\gamma(z) = L_{n_\gamma^{-1}} \circ \gamma(z),$$

where the third equality holds as A is an automorphism and $N/T = \tilde{O}$. This establishes the above equality only on $\tilde{O}$, not on $\tilde{F}$, but both sides of the equality make sense as elements of H/I, and since $\tilde{F}$ is the fixed point set of

I, any two elements of H/I that coincide on $\tilde{O}$ must coincide on $\tilde{F}$. Now the right hand side is independent of x, which implies that the rotational part of γ is independent of x. This means that the bundle $(\tilde{O} \times \mathbb{R}^k)/\pi_1(V)$ is a flat $O(k)$-bundle, and hence so is the normal bundle of F in V. □

9.2. **Normal bundle to a stratum is flat.** Let F_t be the stratum from Section 9.1.

Lemma 9.4. *For all large t, the restriction to O_t of the normal bundle to F_t in H_t/Γ is flat.*

Proof. Let ν_t be the restriction to O_t of the normal bundle to F_t in H_t/Γ. By an obvious contradiction argument it suffices to show that any subsequence of $\{t_i\}$ has a subsequence for which ν_t's are flat; thus passing to subsequences during the proof causes no loss of generality. Since $k_t = \dim(\nu_t)$ can take only finitely many values, we pass to a subsequence for which k_t is constant; we then denote k_t by k. We look at the Grassmanian $G^k(TM)$ of k-planes in the TM with the metric induced by g. Of course, for k-planes tangent to H_t/Γ the metric is also induced by g_t. We fix a point $o_t \in O_t$ and denote by G^k_t the fiber of $G^k(TM)$ over o_t. Let l_t be the fiber of ν_t over o_t.

The fibers of $G^k(TM)$ are (non-canonically) pairwise isometric via the Levi-Civita parallel transport of (M, g). Let $\rho \ll diam(G^k_t)$ be such that the center of mass of [Kar77] is defined in any 4ρ-ball of G^k_t. For large enough t, the ball $B(l_t, 4\rho) \subset G^k_t$ contains no k-plane tangent to F_t because $\rho \ll \mathrm{diam}(G^k_t)$, while l_t, TF_t are h_t-orthogonal and h_t, g_t are ϵ-close, so that the distance between l_t and any k-plane in TF_t is within ϵ of $\mathrm{diam}(G^k_t)$.

Denote by ∇^∞ the connection from Section 4 associated with the parallel transport P^∞. Also denote by ∇_t the Levi-Civita connection of h_t, and let P_t be its parallel transport. The second fundamental form of H_t is uniformly bounded hence P^∞ and P_t are close over any short loop in H_t/Γ.

Since ∇^∞ is flat and compatible with the metric g, it defines the holonomy homomorphism $\phi_t \colon \Gamma \to \mathrm{Iso}(T_{o_t}M)$. Let R_t be the closure of $\phi_t(\Gamma)$ in $\mathrm{Iso}(T_{o_t}M)$. Since R_t is compact, there exists a *finite* subset $S_t \subset \Gamma$ so that for each $r_t \in R_t$ there exists $s \in S_t$ such that $r_t(l_t)$, $\phi_t(s)(l_t)$ are ρ-close in G^k_t. The direction orthogonal to H_t/Γ is ∇^∞-parallel so the R_t-action preserves the subspace $TH_t/\Gamma \subset TM$. Also P^∞ defines an isometry between G^k_t's that is ϕ_t-equivariant and R_t-equivariant. Thus S_t can be chosen independently of t, and we denote S_t by S.

If t is sufficiently large, then for every $s \in S$, the k-planes $\phi_t(s)(l_t)$ and l_t are ρ-close in G^k_t. Indeed, by the exponential convergence of geodesics, if t is large, then s can be represented by a short loop based at o_t. Since the inclusion

$O_t \hookrightarrow H_t/\Gamma$ induces a homotopy equivalence the loop can be assumed to lie on $O_t \subset F_t$. Since F_t is h_t-totally geodesic and l_t is orthogonal to F_t and has complementary dimension, l_t is fixed by P_t along any loop in F_t based at o_t. Hence the same is almost true for P^∞, provided the loop is short enough, which proves the claim since S is finite.

Thus, by the triangle inequality in G_t^k, the k-planes $r_t(l_t)$ and l_t are 2ρ-close for all $r_t \in R_t$. Now let $\bar{l}_t$ be the center of mass of all $r_t(l_t)$'s with $r_t \in R_t$. Clearly, $\bar{l}_t$ is 2ρ-close to l_t, hence $\bar{l}_k$ is transverse to TF_t. Since P^∞ preserves TH_t/Γ, each $r_t(l_t)$, and hence $\bar{l}_t$, is tangent to H_t/Γ. Now we translate $\bar{l}_t$ around O_t using P^∞ along paths of length $\le \mathrm{diam}(O_t) < \epsilon$. Since $\bar{l}_t$ is R_t-invariant and ∇^∞ is flat, this gives a well-defined k-dimensional flat C^0 subbundle $\bar{\nu}_t$ of the restriction of TH_t/Γ to $O_t \subset F_t$.

Note that P_t takes any k-plane tangent to F_t to a k-plane tangent to F_t since F_t is totally geodesic. On the other hand, $\bar{l}_t$ is uniform distance away from any k-plane in TF_t, so its P_t-image remains far away from any k-plane in TF_t. Since on short paths P_t is close to P^∞, we conclude that $\bar{\nu}_t$ is transverse to TF_t for large t. Thus $\bar{\nu}_t$ is C^0 isomorphic to ν_t, so that ν_t carries a C^0 flat connection.

In fact, ν_t also carries a smooth flat connection. Indeed, in the universal cover $\tilde{O}_t$, the C^0 flat connection defines a Γ-equivariant C^0-isomorphism of the pull-back of ν_t to $\tilde{O}_t$ onto $\tilde{O}_t \times \mathbb{R}^k$, where Γ acts as the covering group on the $\tilde{O}_t$-factor and via a holonomy homomorphism on the $\mathbb{R}^k$-factor. This is a smooth action, so the quotient $(\tilde{O}_t \times \mathbb{R}^k)/\Gamma$ is a smooth flat vector bundle that is C^0 isomorphic to ν_t. But any C^0-isomorphic bundles are smoothly isomorphic because the continuous homotopy of classifying maps can be approximated by a smooth homotopy. □

10. Infranilmanifolds are horosphere quotients

Z. Shen constructed in [She94] a pinched negatively curved warped product metric on the product of an arbitrary infranilmanifold and $(0, \infty)$ so that the metric is complete near the ∞-end, but is incomplete at the 0-end. Here we modify Shen's construction to produce a complete pinched negatively curved metric on the product of any infranilmanifold with $\mathbb{R}$.

Let G be a simply-connected nilpotent Lie group acting on itself by left translations, and let K be a compact subgroup of $\mathrm{Aut}(G)$, so that the semidirect product $G \rtimes K$ acts on G by affine transformations. Taking product with the trivial $G \rtimes K$-action on $\mathbb{R}$, we get a $G \rtimes K$-action on $G \times \mathbb{R}$ for which we prove the following.

Theorem 10.1. *$G\times\mathbb{R}$ admits a complete $G\rtimes K$-invariant Riemannian metric of pinched negative curvature. In particular, if N is an infranilmanifold, then $N\times\mathbb{R}$ carries a complete metric of pinched negative curvature.*

Proof. The Lie algebra $L(G)$ can be written as

$$L(G)=L_0\supset L_1\supset\cdots\supset L_k\supset L_{k+1}=0$$

where $L_{i+1}=[L_0,L_i]$. Note that $[L_i,L_j]\subset L_{i+j+1}$. Indeed, assume $i\le j$ and argue by induction on i. The case $i=0$ is obvious and the induction step follows from the Jacobi identity and the induction hypothesis, because $[L_i,L_j]=[[L_0,L_{i-1}],L_j]$ lies in

$$\mathrm{span}([[L_{i-1},L_j],L_0],[[L_0,L_j],L_{i-1}])\subset\mathrm{span}([L_{i+j},L_0],[L_{j+1},L_{i-1}])=L_{i+j+1}$$

The group K preserves each L_i, so we can choose a K-invariant inner product $\langle\ ,\ \rangle_0$ on L. Let

$$F_i=\{X\in L_i\colon\ \langle X,Y\rangle_0=0 \text{ for } Y\in L_{i+i}\}.$$

Then $L=F_0\oplus\cdots\oplus F_k$. Define a new K-invariant inner product $\langle\ ,\ \rangle_r$ on L by $\langle X,Y\rangle_r=h_i(r)^2\langle X,Y\rangle_0$ for $X,Y\in F_i$, and $\langle X,Y\rangle_r=0$ if $X\in F_i$, $Y\in F_j$ for $i\neq j$, where h_i are some positive function defined below. This defines a $G\rtimes K$-invariant Riemannian metric g_r on G.

Let $\alpha_i=i+1$ with $i=0,\cdots,k$ and $a=k+1$. Now define the warping function h_i to be a positive, smooth, strictly convex, decreasing function that is equal to $e^{-\alpha_i r}$ if $r\ge 1$, and is equal to e^{-ar} if $r\le -1$; such a function exists since $a\ge a_i$ for each i. Thus $h_i'<0<h_i''$, and the functions $\frac{h_i'}{h_i}$, $\frac{h_i''}{h_i}$ are uniformly bounded away from 0 and ∞.

Define the warped product metric on $G\times\mathbb{R}$ by $g=s^2g_r+dr^2$, where $s>0$ is a constant; clearly g is a complete $G\rtimes K$-invariant metric. A straightforward tedious computation (mostly done e.g. in [BW]) yields for g-orthonormal vector fields $Y_s\in F_s$ that

$$\langle R_g(Y_i,Y_j)Y_j,Y_i\rangle_g=\tfrac{1}{s^2}\langle R_{g_r}(Y_i,Y_j)Y_j,Y_i\rangle_{g_r}-\tfrac{h_i'h_j'}{h_ih_j},$$

$$\langle R_g(Y_i,Y_j)Y_l,Y_m\rangle_g=\tfrac{1}{s^2}\langle R_{g_r}(Y_i,Y_j)Y_l,Y_m\rangle_{g_r}\quad\text{if } \{i,j\}\neq\{l,m\},$$

$$\langle R_g(Y_i,\tfrac{\partial}{\partial r})\tfrac{\partial}{\partial r}),Y_i\rangle_g=-\tfrac{h_i''}{h_i},\qquad \langle R_g(Y_i,\tfrac{\partial}{\partial r})\tfrac{\partial}{\partial r}),Y_j\rangle_g=0\quad\text{if } i\neq j,$$

$$\langle R_g(\tfrac{\partial}{\partial r},Y_i)Y_j,Y_l\rangle_g=\left(\frac{h_j'}{2h_j}+\frac{h_l'}{2h_l}\right)(\langle[Y_j,Y_i],Y_l\rangle_g+\langle[Y_i,Y_l],Y_j\rangle_g+\langle[Y_j,Y_l],Y_i\rangle_g).$$

Since $[L_i,L_j]\subset L_{i+j+1}$, we have for $Z=\sum_{i=0}^k Z_i$ and $W=\sum_{j=0}^k W_j$ with $Z_i,W_i\in F_i$

$$|[Z,W]|_{g_r}\le\sum_{ij}|[Z_i,W_j]|_{g_r}\le\sum_{ij}\sum_{s>i+j}h_s|[Z_i,W_j]|_{g_0}$$

The above choice of a_i's implies that if $r \geq 1$, then $\sum_{s>i+j} h_s \leq k h_i h_j$. Also $|[Z_i, W_j]|_{g_0} \leq C|Z_i|_{g_0}|W_j|_{g_0}$ where C only depends on the structure constants of L, so that we conclude

$$|[Z, W]|_{g_r} \leq Ck|Z_i|_{g_0}|W_j|_{g_0} \sum_{ij} h_i h_j \leq Ck(k+1)|Z|_{g_r}|W|_{g_r}.$$

It follows that if $r \geq 1$, then the norm of the curvature tensor of g_r is bounded in terms of C, k [CE75, Proposition 3.18]. The same conclusions trivially hold for $r \leq -1$, because then g_r is the rescaling of g_0 by a constant $e^{-ar} > 1$, and also for $r \in [-1, 1]$ by compactness, since g_r is left-invariant and depends continuously of r. Hence $\langle R_g(Y_i, Y_j)Y_l, Y_m\rangle_g \to 0$ as $s \to \infty$.

Also $\langle R_g(\frac{\partial}{\partial r}, Y_i)Y_j, Y_l\rangle_g \to 0$ as $s \to \infty$, because

$$|\langle [Y_j, Y_i], Y_l\rangle_g| = s^2|\langle [Y_j, Y_i], Y_l\rangle_{g_r}| \leq s^2 C(k+1)|Y_j|_{g_r}|Y_i|_{g_r}|Y_l|_{g_r} \leq C(k+1)/s,$$

where the last inequality holds since $s|Y|_{g_r} = 1$ for any g-unit vector Y. It follows that as $s \to \infty$, then R_g uniformly converges to a tensor $\bar{R}$ whose nonzero components are

$$\bar{R}(Y_i, Y_j, Y_j, Y_i) = -\frac{h_i' h_j'}{h_i h_j} \quad \text{and} \quad \bar{R}\left(Y_i, \frac{\partial}{\partial r}, \frac{\partial}{\partial r}, Y_i\right) = -\frac{h_i''}{h_i}.$$

Thus g has pinched negative curvature for all large s. □

Corollary 10.2. *Let E be the total space of a flat Euclidean vector bundle over an infranilmanifold I. Then E is infranil, in particular, $E \times \mathbb{R}$ admits a complete Riemannian metric of pinched negative curvature.*

Proof. Fix a flat Euclidean $\mathbb{R}^k$-bundle over the infranilmanifold I, and write I as G_0/Γ where G_0 a simply-connected nilpotent Lie group and Γ is a discrete cocompact group of affine transformation of G_0 that acts freely. Look at the nilpotent group $G = G_0 \times \mathbb{R}^k$, and let Γ act on the $\mathbb{R}^k$-factor via the holonomy of the flat bundle $\Gamma \cong \pi_1(I) \to O(k)$. Then the infranilmanifold G/Γ is diffeomorphic to the total space of the flat bundle we started with. By Theorem 10.1, $G/\Gamma \times \mathbb{R}$ carries a complete metric of pinched negative curvature. □

11. On geometrically finite manifolds

Proof of Corollary 1.4. Let X/Γ be a geometrically finite pinched negatively curved manifold, let Ω be the domain of discontinuity and Λ be the limit set for the Γ-action at infinity. Let C_ϵ be the ϵ-neighborhood of the convex hull of L. Then C_ϵ/Γ is a codimension zero C^1 submanifold of X/Γ that is homeomorphic to $(X \cup \Omega)/\Gamma$ by pushing along geodesic rays orthogonal to $\partial C_\epsilon/\Gamma$. This homeomorphism restricts to diffeomorphism on the interiors $X/\Gamma \to \mathrm{Int}(C_e)/\Gamma$.

By the discussion in [Bow95, pp263-264], for each end of $(X \cup \Omega)/\Gamma$ there is a parabolic subgroup $\Gamma_z \leq \Gamma$ stabilizing a point $z \in \partial_\infty X$ such that the end has a neighborhood homeomorphic to a neighborhood of the unique end of $(X \cup \partial_\infty X \setminus \{z\})/\Gamma_z$. Again, this homeomorphism restricts to diffeomorphism on the interiors.

By pushing along trajectories of Busemann flow, $(X \cup \partial_\infty X \setminus \{z\})/\Gamma_z$ is homeomorphic to the Γ_z-quotient of a closed horoball H_z centered at z. Note that H_z/Γ_z is a C^2 submanifold of X/Γ_z, and H_z/Γ_z is C^1 diffeomorphic to the product of $[0, \infty)$ and a horosphere quotient, that by Theorem 1.1 is diffeomorphic to the interior of a compact manifold L_z. So H_z/Γ_z is diffeomorphic to the interior of $L_z \times [0,1]$, in which we smooth corners. Compactifying each z-end of C_ϵ/Γ with $L_z \times [0,1]$, we get a compact C^1 manifold whose interior is diffeomorphic to X/Γ. □

Appendix A. Lemmas on nilpotent groups

Lemma A.1. *Given a finitely generated nilpotent group Γ and a positive integer n, there exists a finite subset $F \subset \Gamma$ such that for any $g \in \Gamma$ there is $f \in F$ and $x \in \Gamma$ with $gf = x^n$.*

Proof. We argue by induction on the nilpotency degree of Γ. If Γ is abelian, then the nth powers of elements of Γ form a finite index subgroup, and we can take F to the set of coset representatives of this subgroup. In general, if Z denotes the center of Γ, then by induction the result is true for Γ/Z for the finite subset $\{aZ : a \in F_1\}$ of G/Z, where F_1 is some finite subset of Γ. Thus, an arbitrary $g \in \Gamma$ satisfies $gf_1 = x^n z$ for some $f_1 \in F_1$, $x \in \Gamma$, $z \in Z$. Again since Z is abelian, the set Z^n of nth powers is a finite index subgroup of Z. Let F_2 be a set of coset representatives of Z^n in Z so that $z = y^n f_2$ for some $y \in Z$, $f_2 \in F_2$. Then $gf_1 = x^n y^n f_2 = (xy)^n f_2$, so $gf_1 f_2^{-1} = (xy)^n$, and the assertion holds for Γ with $F = F_1 F_2^{-1}$. □

Lemma A.2. *Let Γ be a finitely generated nilpotent group. Then there exists a finite set $F \subset \Gamma$ such that for any $x \in \Gamma$, $g \in [\Gamma, \Gamma]$ there is $h \in [\Gamma, \Gamma]$, $f \in F$ with $x^2 g = (xh)^2 f$.*

Proof. As usual we denote $\Gamma_1 = \Gamma$, $\Gamma_{i+1} = [\Gamma, \Gamma_i]$, so that the nilpotency degree k of Γ is the largest integer for which Γ_k is nontrivial. Since $[\Gamma, \Gamma_k]$ is trivial, Γ_k lies in the center of Γ. We argue by induction on k. The case $k = 1$, i.e. when Γ is abelian, is obvious for $F = \{1\}$. If Γ is of nilpotency degree $k > 1$, then by induction the statement is true in Γ/Γ_k. Let $F_1 \subset \Gamma$ be the set of coset representatives of the corresponding finite set for Γ/Γ_k, so that given $x \in \Gamma$, there exists $z \in \Gamma_k$, $g, h \in [\Gamma, \Gamma]$, $f_1 \in F_1$ with $x^2 g = (xh)^2 f_1 z$. By

Lemma A.1 applied to Γ_k, we get $z = y^2 f_2$ for some $y \in \Gamma_k$, $f_2 \in F_2$ where F_2 is a finite subset of Γ_k. Then $x^2 g = (xh)^2 f_1 z = (xyh)^2 f_1 f_2$, and since $yh \in [\Gamma, \Gamma]$ the proof is complete. □

Lemma A.3. *Let C be a maximal compact subgroup of a connected nilpotent Lie group N. Then C is equal to the unique maximal compact subgroup of the center of N, in particular C is a torus.*

Proof. Any maximal compact subgroup C of N lies in the center Z of N [Iwa45]. Hence C is also maximal compact in Z. A maximal compact subgroup is homotopy equivalent to the ambient group, hence since N is connected, so is Z and C. Now Z connected abelian, hence Z is isomorphic to the product of a real vector space and a torus, so C equals to the torus. □

Appendix B. Isometries are smooth

Proposition B.1. *Let X be a smooth manifold equipped with a complete C^0-Riemannian metric of curvature bounded above and below in the comparison sense. Then the isometry group acts on X by C^3-diffeomorphisms.*

Proof. The isometry group of any complete locally-compact metric space is locally compact [KN96]. By [MZ55, Chapter 5], any locally compact subgroup of $\mathrm{Diffeo}^r(X)$ with $r > 0$ is a Lie group and the action is C^r. Thus, it suffices to show that each individual isometry is C^3. The construction of harmonic coordinates in [Nik83] starts with the C^0-distance coordinates $(d(x, a_1), ..., d(x, a_n))$ at $x \in X$, where the geodesic segments $[x, a_i]$ are pairwise orthogonal, and then solves the Dirichlet problem in a small ball around x with values on the boundary sphere given by $d(x, a_i)$. The solutions are the so-called harmonic coordinates. Their transition functions are $C^{3,\alpha}$ (and the metric tensor in this coordinates is $C^{1,\alpha}$ even though we do not need this fact here). This construction is clearly invariant under isometries, so any isometry has the same smoothness in harmonic coordinates as the identity map, namely $C^{3,\alpha}$. □

Appendix C. Local formula of Ballmann and Brüning

Let X be a simply-connected manifold of pinched negative curvature. Fix a point at infinity of X, and let T be the unit vector field tangent to the Busemann flow $b_t(x)$ towards that point. For a curve $\alpha(s)$ in X, denote $\alpha(0) = x$ and $\alpha'(0) = u$. Look at the 1-parameter family of geodesic rays $\alpha(s, t) = b_t(\alpha(s))$ and the corresponding family of Jacobi fields J. Let $v, w \in$

T_xX and let $X(t,s)$, $Y(t,s)$ be vector fields along $a(s,t)$ such that $X(0,0)=v$, $Y(0,0)=w$ and $\nabla_T X = \nabla_T Y = 0$. Then one defines a tensor field $\bar{S}$ by

$$\langle \bar{S}(u,v), w\rangle = -\int_0^\infty \langle R(T,J)X,Y\rangle(t,0)dt$$

Ballmann and Brüning [BB01] define a new connection $\bar{\nabla}$ by $\bar{\nabla} = \nabla - \bar{S}$, and show that $\bar{\nabla}$ is a C^0 flat connection that is compatible with the metric, and that satisfies $\bar{\nabla}T = 0$ and

$$|\bar{\nabla}_X Y - \nabla_X Y| \le C(a)|X||Y| \text{ for any } X,Y \tag{C.1}$$

Since $\bar{\nabla}$ is flat, for any $x,y\in M$ we have a well defined parallel transport with respect to $\bar{\nabla}$ from x to y which we denote by $P^{\bar{\nabla}}_{xy}$.

Lemma C.2. *For any $x,y\in M$ parallel transport through infinity P^∞_{xy} coincides with $P^{\bar{\nabla}}_{xy}$.*

Proof. First suppose x, y lie in the same horosphere H_{t_0}. Consider the quadrangle $xb_t(x)b_t(y)y$. By flatness $P^{\bar{\nabla}}_{xy}$ is equal to the $\bar{\nabla}$ parallel transport along three other sides of this quadrangle $P^{\bar{\nabla}}_{b_t(y)y}\circ P^{\bar{\nabla}}_{b_t(x)b_t(y)}\circ P^{\bar{\nabla}}_{xb_t(x)}$. Also if α is a trajectory of the Busemann flow, then $J=T$ so that $\bar{S}=0$, therefore parallel transports along Busemann trajectories coincide with Levi-Civita parallel transports. Thus

$$P^{\bar{\nabla}}_{xy} = P_{b_t(y)y}\circ P^{\bar{\nabla}}_{b_t(x)b_t(y)}\circ P_{xb_t(x)}$$

Since $d(b_t(x)b_t(y))\to 0$ as $t\to\infty$, by (C.1) we have that $P^{\bar{\nabla}}_{b_t(x)b_t(y)}$ becomes arbitrary close to $P_{b_t(x)b_t(y)}$ for large t and therefore

$$P^{\bar{\nabla}}_{xy} = \lim_{t\to\infty} P_{b_t(y)y}\circ P_{b_t(x)b_t(y)}\circ P_{xb_t(x)} \tag{C.3}$$

Similarly, by construction P^∞ commutes with Busemann flow and hence

$$P^{\infty}_{xy} = P_{b_t(y)y}\circ P^{\infty}_{b_t(x)b_t(y)}\circ P_{xb_t(x)}$$

As before $P^\infty_{b_t(x)b_t(y)}$ becomes arbitrary close to $P_{b_t(x)b_t(y)}$ for large t and therefore

$$P^{\bar{\nabla}}_{xy} = \lim_{t\to\infty} P_{b_t(y)y}\circ P_{b_t(x)b_t(y)}\circ P_{xb_t(x)} \tag{C.4}$$

Comparing (C.3) and (C.4) we conclude that $P^\infty_{xy} = P^{\bar{\nabla}}_{xy}$ for any x, y in the same horosphere.

Finally, since $\bar{\nabla}$ and ∇^∞ are flat, and their parallel transports coincide with the Levi-Civita parallel transport along trajectories of the Busemann flow $P^\infty_{xy} = P^{\bar{\nabla}}_{xy}$ for any x, y. $\square$

Appendix D. Concave functions and submetries on Alexandrov spaces

The proof of the following lemma is due to A. Petrunin.

Lemma D.1. *Let X be an Alexandrov space of curv $\geq k$ with $\partial X = \emptyset$. Let $f\colon X \to \mathbb{R}$ be a Lipschitz function with a local maximum at q. Suppose f is strictly concave on an open set U containing q. Then $d(\cdot, q)$ has no critical points on $U \backslash \{q\}$.*

Proof. Let $x \in U, x \neq q$. By [PP] $\nabla f(x)$ is defined to be equal to $v \in T_x X$ if $df(v) = |v|^2$ and $\frac{df(u)}{|u|}$ attains a positive maximum at v. Since x is not a point of maximum of f, $\nabla f(x) \neq 0$ and $|\nabla f(z)| \geq c > 0$ for all z near x. Consider the gradient flow for f as defined in [PP]. Consider a gradient line $\gamma(t)$ passing through x so that $\gamma(0) = x$. By [PP] the curve $\gamma(t)$ is locally Lipschitz.

We claim that $\gamma(t)$ can be extended to be a gradient line defined on $(-\epsilon, \infty)$ for some $\epsilon > 0$. Indeed, suppose this is not true. By [PP, Lemma 3.2.1(a)] the gradient flow of *any* concave function is 1-Lipschitz, so it defines a deformation retraction of any superlevel set $\{f \geq c\}$ contained in U onto q and hence $\{f \geq c\}$ is contractible (for a different proof see also [Kap02, Lemma 5.2]). Take $c = f(x)$ and let $\epsilon > 0$ be small enough so that $\{f \geq c - \epsilon\}$ is contained in U. Since γ can not be extended backwards beyond zero, the gradient flow gives a deformation retraction of $\{f \geq c - \epsilon\} \backslash \{x\}$ onto q. To see that this is impossible we prove that $H^{n-1}(\{f \geq c - \epsilon\} \backslash \{x\}, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Indeed, since $\{f \geq c - \epsilon\}$ is contractible, from the long exact cohomology sequence of a pair we see that

$$H^{n-1}(\{f \geq c - \epsilon\} \backslash \{x\}, \mathbb{Z}_2) \cong H^n(\{f \geq c - \epsilon\}, \{f \geq c - \epsilon\} \backslash \{x\}, \mathbb{Z}_2).$$

By [Per93], for some very small $\delta > 0$, the ball $B(x, \delta)$ is contractible and is contained in $\{f \geq c - \epsilon\}$, and $d(\cdot, x)$ has no critical points in $B(x, 2\delta) \backslash \{x\}$. Therefore, by excision

$$H^n(\{f \geq c - \epsilon\}, \{f \geq c - \epsilon\} \backslash \{x\}, \mathbb{Z}_2) \cong H^n(B(x, \delta), B(x, \delta) \backslash \{x\}, \mathbb{Z}_2)$$

which is isomorphic to $H^{n-1}(S(x, \delta), \mathbb{Z}_2) \cong \mathbb{Z}_2$, where the last equality holds since by [Per93], $S(x, \delta)$ is homotopy equivalent to the Alexandrov space $\Sigma_x X$ with $\partial \Sigma_x X = \emptyset$, and since any Alexandov space without boundary has top-dimensional $\mathbb{Z}_2$-cohomology isomorphic to $\mathbb{Z}_2$ [GP93].

Let v be any *left* tangent of γ at 0. Here following [PP] we say that $v \in T_x X$ is a left tangent vector if $v = \lim_{i \to \infty} \frac{1}{|t_i|} exp_x^{-1} \gamma(t_i)$ for some sequence $t_i \to 0^-$. (One should think of v as $-\nabla f(x)$). By above $f(\gamma(t)) - ct$ is nondecreasing for small t. Therefore $v \neq 0$. We claim that $\angle uv \geq \pi/2$ for any direction u

of a shortest geodesic from x to y with $f(y) > f(x)$. Indeed, by [PP, Lemma 3.2.1(a)] the gradient flow of *any* concave function is 1-Lipschitz. Consider a cutoff concave function $\hat{f}(\cdot) = \min\{f(\cdot), f(y)\}$. Clearly the forward gradient flow of $\hat{f}$ fixes y and coincides with the gradient flow of f near x. Since the gradient flow of $\hat{f}$ is 1-Lipschitz, $d(y, \gamma(t))$ is nonincreasing near $t = 0$. By the first variation formula, this implies that $\angle uv \geq \pi/2$. In particular, this is true for $y = q$. Since any shortest from x to q points strictly inside the convex set $\{f \geq f(x)\}$ this inequality is in fact strict, i.e $\angle uv > \pi/2$. □

Definition D.2. *A map $f\colon X \to Y$ between two metric spaces is called a submetry if for any $x \in X$ and any $r > 0$ one has $f(B_r(x)) = B_r(f(x))$.*

We need some properties of submetries collected below.

Lemma D.3. *Suppose X is an Alexandrov space of curv $\geq k$ and $f\colon X \to Y$ is a submetry. Then*

a) Y is an Alexandrov space of curv $\geq k$ [BGP92] .
b) For any $x \in X, y \in Y$ we have $d(x, f^{-1}(y)) = d(f(x), y)$.
c) For any $x \in X$ and any shortest geodesic $\gamma\colon [0,1] \to Y$ with $\gamma(0) = f(x)$ there exists a shortest geodesic $\tilde{\gamma}\colon [0,1] \to Y$ with $\tilde{\gamma}(0) = x$ such that $f(\tilde{\gamma}(t)) = \gamma(t)$. The geodesic $\tilde{\gamma}$ is called a horizontal lift of γ. Moreover, if $\gamma(t)$ can be extended to a shortest $\gamma\colon [-\epsilon, 1] \to Y$ then the horizontal lift is unique.

Moreover the statement a) is true locally in the following sense: if $B(p,1) \subset U \subset X$ and $g\colon U \to Y$ is a submetry and diam$(g^{-1}(y)) < 1/10$ for any $y \in Y$, then for any triangle with vertices in $B(g(p), 1/4)$ the Toponogov comparison with curv $\geq k$ holds.

The proofs of parts b) and c) are elementary and left to the reader (see [Lyt] for details).

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